Limits

Warm-Up

- Find f(2) for each function:

1. $f(x) = \frac{4x^2-5}{x^3+4}$
2. $f(x) = \sin(\pi \frac{x}{2})$
3. $f(x) = \begin{cases} 3x - 1\,, x < 2 \\ \frac{1}{x^2-1}\,, x \geq 2 \end{cases}$

- Write in reduced form:

$$\frac{2x^2 - x}{2x^2 + x - 1}$$

Learning targets.   I can…

- Find one- and two-sided limits

Why limits?

- Average rate of change and instantaneous rate of change are two very important ideas in calculus. We're going to talk about them using a physics problem and see how it ties to the idea of a limit.
- A moving body's **average speed** during an interval of time is found by dividing the distance covered by the elapsed time.
- How could we determine the average speed of the marble on the track.

Why limits?

- The marble's position as a function of time is given by $y= 1.7\ t^2$
- The average speed over any interval is given by the distance traveled, Δy, divided by the length of the interval, Δx.
- Calculate the average speed from t=0 to t=2 seconds
- Average speed = $\frac{\Delta y}{\Delta x} = \frac{f(2)-f(0)}{2-0}$

Why limits?

- How would we find the **instantaneous speed** of the marble? (i.e., the speed of the marble at a given instant in time?)
- Find the speed of the marble at t=2s
- We can calculate the average speed of the marble *over a very small interval of time*, from t=2 to t=2+h
- $\frac{\Delta y}{\Delta x} = \frac{f(2+h)-f(2)}{(2+h)-2} = \frac{1.7(2+h)^2-1.7(2)^2}{h}$

Why limits?

- $\frac{\Delta y}{\Delta x} = \frac{f(2+h)-f(2)}{(2+h)-2} = \frac{1.7(2+h)^2-1.7(2)^2}{h}$
- We can't use this formula to calculate the speed at the exact instant t=2 (we would need h=0) because that would result in division by 0, which is undefined.
- BUT, we can get a good idea of what is happening at t=2 by evaluating the formula for values of h very *close* to 0.

Why limits?

- $\frac{\Delta y}{\Delta x} = \frac{f(2+h)-f(2)}{(2+h)-2} = \frac{1.7(2+h)^2 - 1.7(2)^2}{h}$
- Evaluate the formula for the following values of h:

Length of time interval, h	Average speed for interval, Δy/Δx
1	8.5
0.1	6.97
0.01	6.817
0.001	6.8017
0.0001	6.80017
0.00001	6.800017

Why limits?

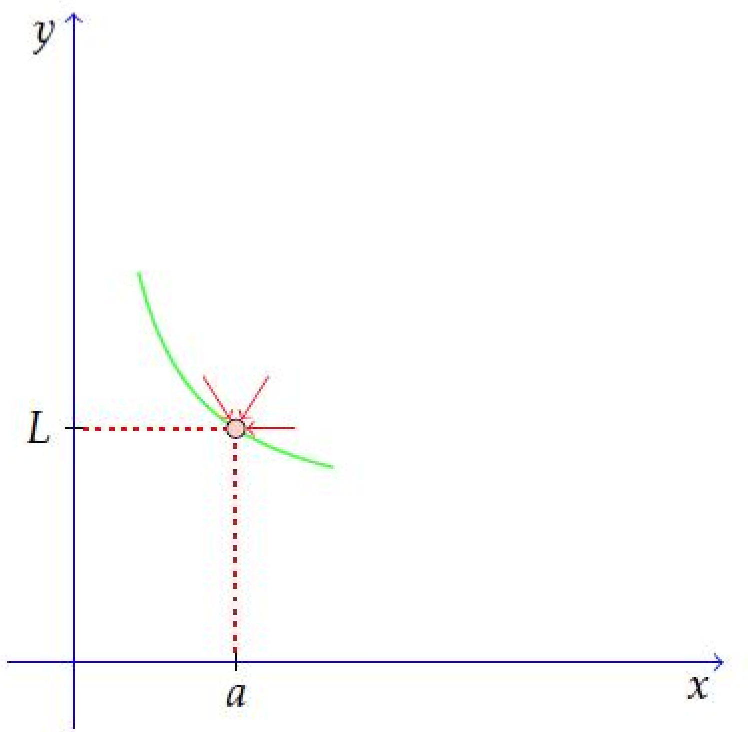
- $\frac{\Delta y}{\Delta x} = \frac{f(2+h)-f(2)}{(2+h)-2} = \frac{1.7(2+h)^2-1.7(2)^2}{h}$
- As h gets closer and closer to 0, the value of the function gets closer and closer to a limiting value – 6.8.
- In other words, we are getting closer and closer to the speed at a particular instant of time (2 seconds).

Why limits?

- $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-x}$
- This expression is called the **difference quotient**
- The difference quotient is a measure of the *average rate of change* of the function over an interval. It is the slope of a secant line.
- The **limit** of the difference quotient as h→0 is the *instantaneous rate of change*. The slope of the tangent line at the point of interest.

Definition of Limit

- Limits let us describe how the outputs of a function behave as the inputs approach some particular value.
- the limit of a function is what the function "approaches" when the input (the variable "x" in most cases) approaches a specific value.



Definition of Limit

- Let c and L be real numbers. The function **f has limit L as x approaches c** if, given any positive number ε, there is a positive number δ such that for all x

0 < |x-c| < δ ➔ |f(x)-L| < ε

We write

$$\lim_{x \to c} f(x) = L$$

Read "the limit of f of x as x approaches c equals L

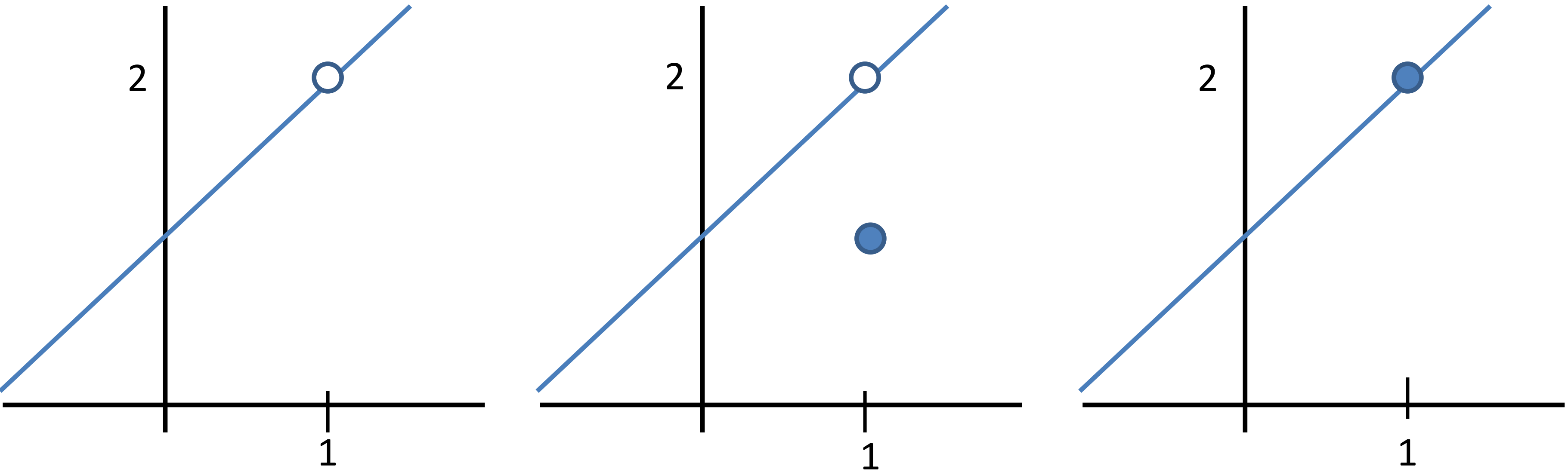
Definition of Limit

For a (two-sided) **limit** to exist at a specific input value, the *left* and *right-hand* limits must be equal.

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Note: The output value of the function, f(c), may or may not equal the limit (or even exist at all!).

Example: Limit



$f(x) = \frac{x^2-1}{x-1}$

$g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

h(x) = x+1

- Note that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = \lim_{x\to 1} h(x) = 2$ even though f(1) ≠ 2 and g(1) ≠ 2.

Determining if a limit exists:

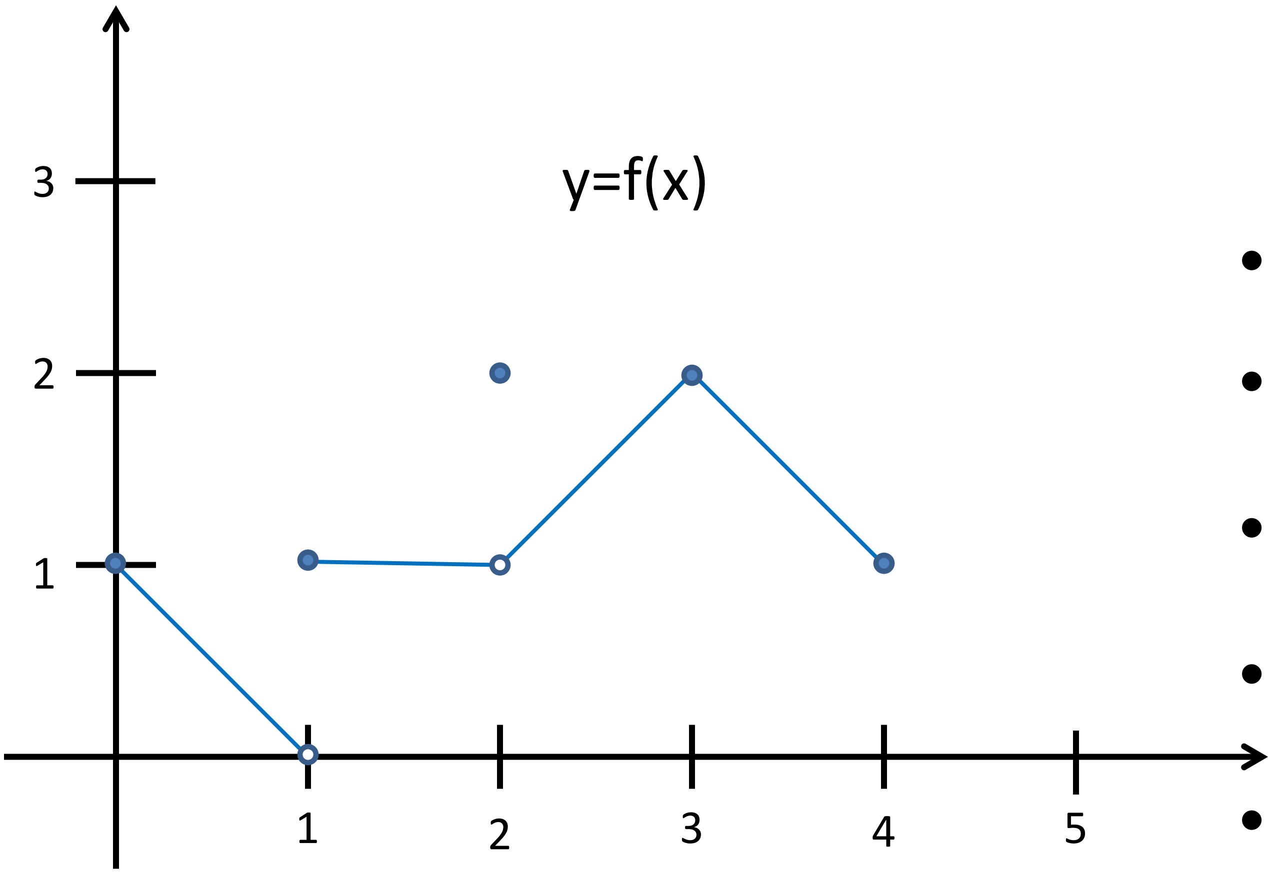
- First, try direct substitution!
- $\lim_{x \to -3} (2x + 5) \quad =$
- Verify the limit graphically (the limit as $x \to -3^-$ must equal the limit as $x \to -3^+$)

Determining if a limit exists:

- Find:
- $\lim_{x \to 0} \left(\frac{sinx}{x}\right) \quad =$
- We can’t use direct substitution because we get 0/0, which is undefined.
- Let’s make a table of values

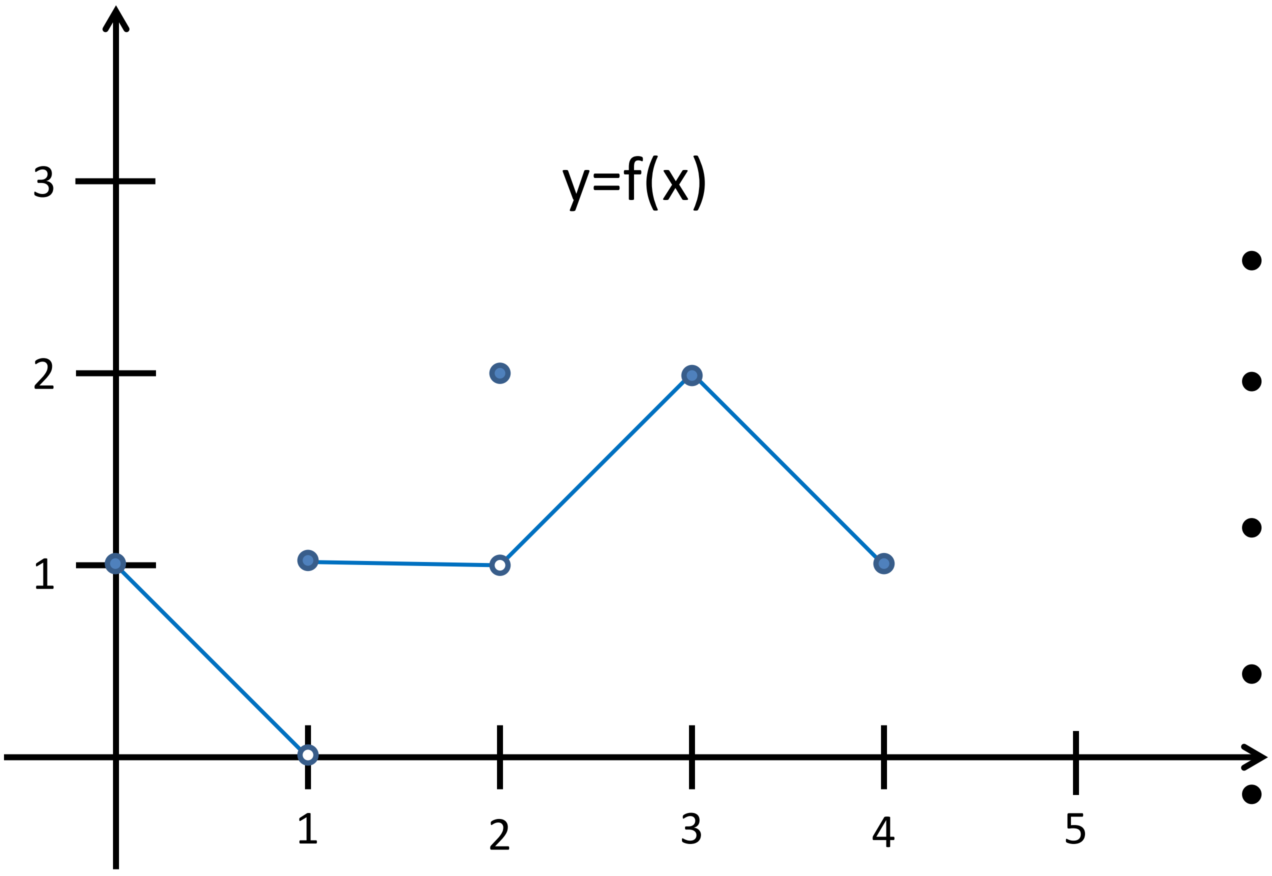
Determining if a limit exists:

- Find:
- $\lim_{x \to 0} (\frac{sinx}{x}) \quad =$
- Or we can graph the function to see if the limit exists (remember, the left-hand limit must equal the right-hand limit in order for the limit to exist).



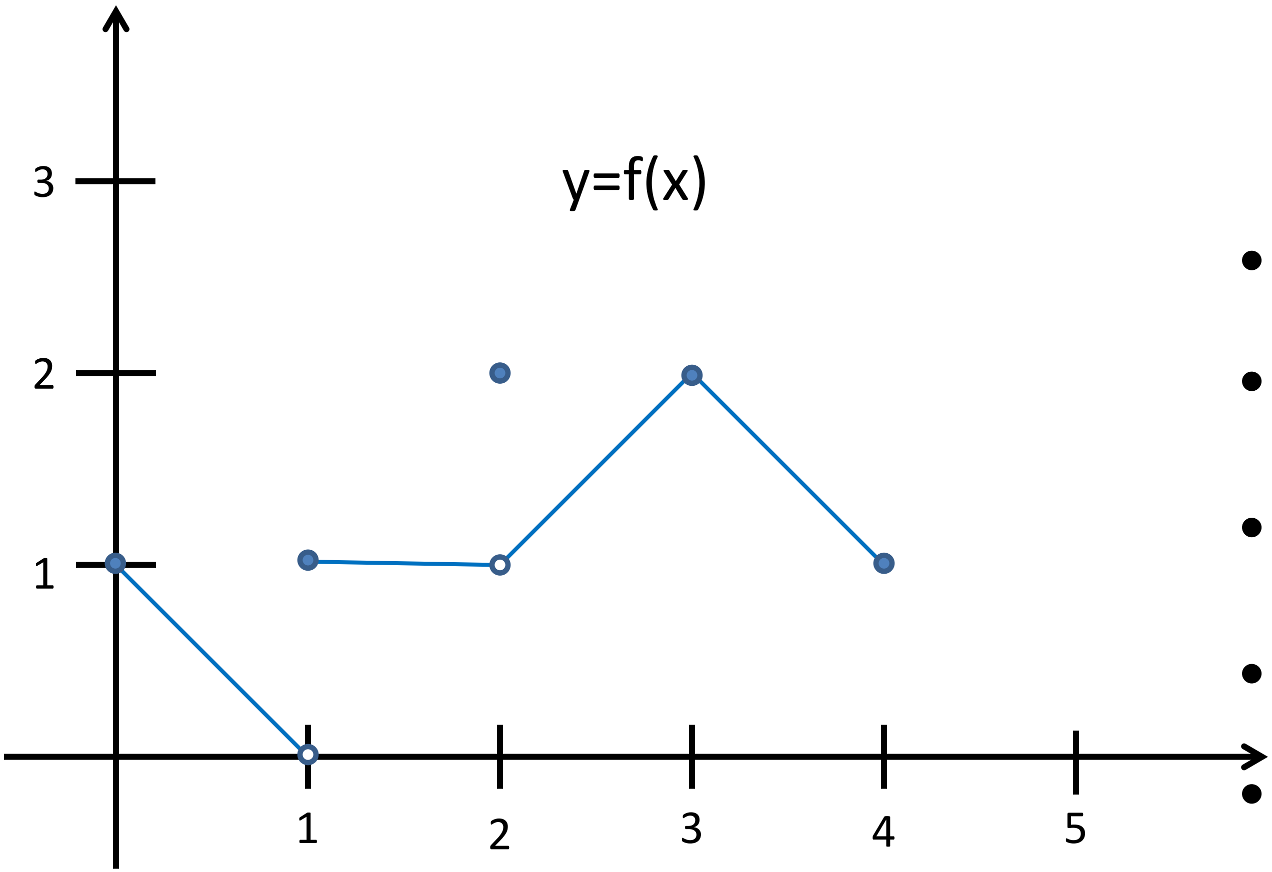
- Find:
- $\lim_{x\to 0+} f(x) = 1$
- $\lim_{x\to 1-} f(x) = 0$
- $\lim_{x\to 1+} f(x) = 1$
- $\lim_{x\to 1} f(x)\ DNE$

$$f(x)=\begin{cases} -x+1\ , & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x-1, & 2 < x \le 3 \\ -x+5, & 3 < x \le 4 \end{cases}$$



- Find:
- $\lim_{x \to 2-} f(x) = 1$
- $\lim_{x \to 2+} f(x) = 1$
- $\lim_{x \to 2} f(x) = 1$
- f(2) =2

$$f(x)=\begin{cases} -x+1\ , & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & x = 2 \\ x-1, & 2 < x \leq 3 \\ -x+5, & 3 < x \leq 4 \end{cases}$$



$$f(x)=\begin{cases} -x+1\ , & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x-1, & 2 < x \le 3 \\ -x+5, & 3 < x \le 4 \end{cases}$$

- Find:
- $\lim_{x\to 3-} f(x) = 2$
- $\lim_{x\to 3+} f(x) = 2$
- $\lim_{x\to 3} f(x) = 2$
- f(3)=2
- $\lim_{x\to 4-} f(x) = 1$

Assignment

- P. 62 # 1-6, 17-20 (graph by hand!), 31, 32